Solving Cubic Equations Using a Triangle

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Introduction

In the 1500's, new breakthroughs allowed mathematicians to solve the cubic equation, which had remained unsolved for thousands of years.



However, in the case of 3 real roots, the formulas yielded square roots of negative numbers, which mathematicians did not know how to handle at the time.

Then, François Viète derived an alternative solution with only real numbers using just a triangle, a circle and a little trigonometry.

Depressing the Cubic Equation

$$ax^3 + bx^2 + cx + d = 0$$

The General Cubic Equation

Often, it is more convenient to use a simplified form, known as the *depressed cubic.*

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

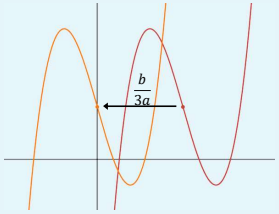
$$\left(w - \tfrac{b}{3a}\right)^3 + \tfrac{b}{a}\left(w - \tfrac{b}{3a}\right)^2 + \tfrac{c}{a}\left(w - \tfrac{b}{3a}\right) + \tfrac{d}{a} = 0 \qquad \textit{Substitute}\ x = w - \tfrac{b}{3a}$$

$$w^3 - \frac{b}{a}w^2 + \frac{b^2}{3a^2}w - \frac{b^3}{27a^3} + \frac{b}{a}w^2 - \frac{2b^2}{3a^2}w + \frac{b^3}{9a^3} + \frac{c}{a}w - \frac{bc}{3a^2} + \frac{d}{a} = 0$$

$$w^3 + \left(-\frac{b^2}{3a^2} + \frac{c}{a}\right)w + \left(\frac{2b^3}{27} - \frac{bc}{3a^2} + \frac{d}{a}\right) = 0 \qquad \textit{Substitute}\ p = -\frac{b^2}{3a^2} + \frac{c}{a},\ q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}$$

$$w^3 + \underbrace{p}\ w + \underbrace{q} = 0$$

The coordinate of a cubic equation's inflection point is $\frac{b}{3a}$, so by this substitution, we are shifting the cubic left or right such that the inflection point lies on the y-axis, as is the case for all depressed cubics. This shifts the roots by the same distance as well.



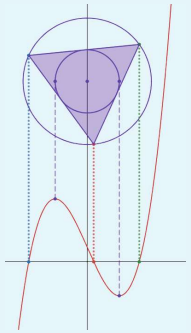
After we find the roots of the depressed cubic, we simply shift the roots back $\frac{b}{3a}$ units by substituting back into $x = w - \frac{b}{3a}$

Setting up the Triangle

From the graph of any cubic equation, it is always possible to draw an equilateral triangle above the cubic such that:

1. Its center has the same x-coordinate as the inflection point
2. The vertices of the triangle line up with the roots of the cubic
3. The leftmost and rightmost points of the incircle of the triangle line up with the extrema of the cubic

Using these 3 conditions, it is possible to find the exact locations of all 3 roots.



First, we know that for all depressed cubics, the inflection point lies on the y-axis, so the triangle must have a center on the y-axis.

Sizing the Triangle

To find the size of the incircle and the triangle, we can use condition 3.

The extrema occur when the derivative of the graph is zero.

The original equation is $w^3 + pw + q = 0$

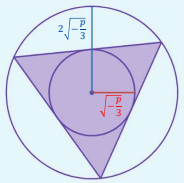
The derivative is $3w^2 + p$, so we set it equal to 0

$$3w^2 + p = 0$$
$$w^2 = -\frac{p}{3}$$
$$w = \pm\sqrt{-\frac{p}{3}}$$

There are two solutions, corresponding to the two extrema. There is a negative sign attached to p, but p is always negative in the case of 3 real roots, so we always obtain two real numbers.

Now, the radius of the incircle must be $\sqrt{-\frac{p}{3}}$, because its leftmost and rightmost points have *x* coordinates $+\sqrt{-\frac{p}{3}}$ and $-\sqrt{-\frac{p}{3}}$.

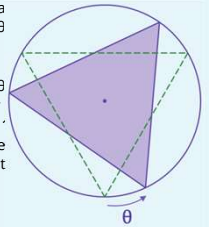
What we really want is the radius of the circumcircle of the triangle, to compute rotations. For an equilateral triangle, the radius of the circumcircle is twice the radius of the incircle, thus the circumcircle's radius (r) is $2\sqrt{-\frac{p}{3}}$.



$$r = 2\sqrt{-\frac{p}{3}}$$

Finding Coordinates of the Vertices

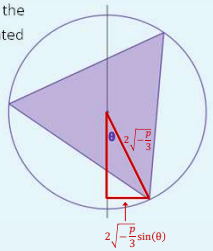
The initial downward pointing triangle has a vertex on the y-axis, and it is rotated by θ radians to correspond with the roots.



These vertices are rotated by an angle θ around the circumcircle, with radius $2\sqrt{-\frac{p}{3}}$. Therefore, if we look at the points on the circumcircle, the x-coordinate of the first root is $2\sqrt{-\frac{p}{3}}\sin(\theta)$.

The other points have initial angles on the triangle of $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, so their full rotated angles are $\theta + \frac{2\pi}{3}$ and $\theta + \frac{4\pi}{3}$.
Thus, the vertex coordinates are:

$$\begin{cases} x_1 = 2\sqrt{-\frac{p}{3}}\sin(\theta) \\ x_2 = 2\sqrt{-\frac{p}{3}}\sin\left(\theta + \frac{2\pi}{3}\right) \\ x_3 = 2\sqrt{-\frac{p}{3}}\sin\left(\theta + \frac{4\pi}{3}\right) \end{cases}$$



Finding θ

The original cubic equation in factored form is

$$(x - x_1)(x - x_2)(x - x_3) = 0$$

with x_1, x_2 and x_3 as roots

Multiply out

$$x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_2x_3 + x_1x_3)x - x_1x_2x_3 = 0$$

Matching coefficients with the original depressed cubic equation, we get 3 equations known as the symmetric equations:

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1x_2 + x_2x_3 + x_1x_3 = p \\ -x_1x_2x_3 = q \end{cases}$$

Using the formulas for each root in terms of θ, we can plug them into the 3rd symmetric equation to isolate θ.

$$-\left(2\sqrt{-\frac{p}{3}}\sin(\theta)\right)\left(2\sqrt{-\frac{p}{3}}\sin\left(\theta + \frac{2\pi}{3}\right)\right)\left(2\sqrt{-\frac{p}{3}}\sin\left(\theta + \frac{4\pi}{3}\right)\right) = q$$

$$\frac{8p}{3}\sqrt{-\frac{p}{3}}\sin(\theta)\sin\left(\theta + \frac{2\pi}{3}\right)\sin\left(\theta + \frac{4\pi}{3}\right) = q$$

Use a trigonometric identity

$$\sin(\theta)\sin\left(\theta + \frac{2\pi}{3}\right)\sin\left(\theta + \frac{4\pi}{3}\right) = -\frac{1}{4}\sin(3\theta)$$

$$-\frac{2p}{3}\sqrt{-\frac{p}{3}}\sin(3\theta) = q$$

Move all constants to the right side

$$\sin(3\theta) = -\frac{3q}{2p}\sqrt{-\frac{3}{p}}$$

The expression on the right side will always be in the interval $[-1,1]$
So, we can take arcsine

$$3\theta = \arcsin\left(-\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)$$

Solve

$$\theta = \frac{1}{3}\arcsin\left(-\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)$$

Computing the Roots

Now that we have both **r** and **θ**, we can plug them into our root formulas to finally obtain all 3 roots of our depressed cubic equation:

$$\left.\begin{aligned} x_1 &= 2\sqrt{-\tfrac{p}{3}}\sin\left(\tfrac{1}{3}\arcsin\left(-\tfrac{3q}{2p}\sqrt{-\tfrac{3}{p}}\right)\right) \\ x_2 &= 2\sqrt{-\tfrac{p}{3}}\sin\left(\tfrac{1}{3}\arcsin\left(-\tfrac{3q}{2p}\sqrt{-\tfrac{3}{p}}\right) + \tfrac{2\pi}{3}\right) \\ x_3 &= 2\sqrt{-\tfrac{p}{3}}\sin\left(\tfrac{1}{3}\arcsin\left(-\tfrac{3q}{2p}\sqrt{-\tfrac{3}{p}}\right) + \tfrac{4\pi}{3}\right) \end{aligned}\right\} x_{1,2,3} = 2\sqrt{-\tfrac{p}{3}}\sin\left(\tfrac{1}{3}\arcsin\left(-\tfrac{3q}{2p}\sqrt{-\tfrac{3}{p}}\right) + \tfrac{2\pi k}{3}\right)$$

Where $k = \{0,1,2\}$

To get back to the general cubic equation from the depressed form, we simply use the previous formulas for *p* and *q*, then shift the roots by subtracting $\frac{b}{3a}$ from each.

$$x_{1,2,3} = 2\sqrt{-\frac{\left(-\frac{b^2}{3a^2} + \frac{c}{a}\right)}{3}}\sin\left(\frac{1}{3}\arcsin\left(-\frac{3\left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right)}{2\left(-\frac{b^2}{3a^2} + \frac{c}{a}\right)}\sqrt{-\frac{3}{\left(-\frac{b^2}{3a^2} + \frac{c}{a}\right)}}\right) + \frac{2\pi k}{3}\right) - \frac{b}{3a}$$

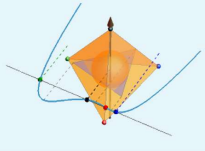
And our final answer is:

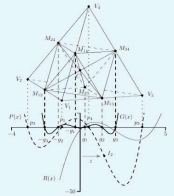
$$x_{1,2,3} = 2\sqrt{\frac{b^2}{9a^2} - \frac{c}{3a}}\sin\left(\frac{1}{3}\arcsin\left(\frac{\left(\frac{b^3}{27} - \frac{bc}{6a^2} + \frac{d}{2a}\right)}{\left(\frac{b^2}{9a^2} - \frac{c}{3a}\right)}\sqrt{\frac{1}{\left(\frac{b^2}{9a^2} - \frac{c}{3a}\right)}}\right) + \frac{2\pi k}{3}\right) - \frac{b}{3a}$$

Where $k = \{0,1,2\}$

Extension to Higher Dimensions

Just as we can construct a triangle to solve a cubic, the quartic equation can be solved by constructing a tetrahedron in 3-D space. Similar to how the vertices of the triangle line up with the roots of the cubic, it is always possible to rotate and scale a tetrahedron such that, if it is floating "above" the Cartesian plane, all 4 vertices have the same x-coordinates as the roots.





Acknowledgements

Main Presentation

Wikipedia Article on Cubic Equations
en.wikipedia.org/wiki/Cubic_equation#Trigonometric_and_hyperbolic_solutions
Video by MichaelMaths about the Triangular Solution
youtube.com/watch?v=aAYTk-VIKQI&t=1217s
Identity based derivation of the solution
youtube.com/watch?v=K6SWIr1dEHQ

Extension

Wolfram Interactive
demonstrations.wolfram.com/GeometryOfQuarticPolynomials/
I used this interactive to form the image on the left.

Interesting paper on the topic
nickalls.org/dick/papers/maths/tetrahedron2012.pdf
I obtained the image on the right from this paper