The Geometry of Polynomials

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Introduction

Often, when algebraic equations are explored, they are purely confined to the language of symbols and expressions. However, there exist startling ways to express polynomials in the language of geometry: in terms of shapes and angles. In this presentation, I will provide 2 examples of a geometric lens applied to polynomial equations.

The first example involves cubic equations, whose methods of solving are rarely taught in algebra classrooms. However, there is an exciting connection between the mechanics of complex numbers and a triangle rotation in the coordinate plane.

The second example concerns quartic equations, which are even more miraculous than the preceding. Their solutions are encoded in the 3-dimensional analogue of a triangle: a tetrahedron. There exists a similar but even more shocking relation between the midpoints of a tetrahedron and the solutions of an auxiliary cubic equation.

Depressing the Cubic and Quartic Equations

When analyzing cubic and quartic equations, it is easier to consider a simplified form known as the *Depressed Form.*

The Depressed Form has the 2nd highest term removed:

Depressed cubic form: $x^3 + px + q = 0$ (no x^2 term)

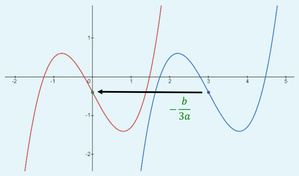
Depressed quartic form: $x^4 + px^2 + qx + s = 0$ (no x^3 term)

All depressed cubics have the inflection point on the y-axis.

The x-coordinate of the inflection point is at $x_i = -\frac{b}{3a}$,

so we can shift over by $-\frac{b}{3a}$

by substituting $x = x' - \frac{b}{3a}$



A similar shift can be done for a quartic equation, except now we must consider a higher order inflection point.

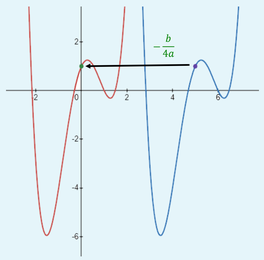
I will call the points where the 3rd derivative is 0 "aberrancy points" because the 3rd derivative can be interpreted as the aberrancy of a function.

All depressed quartics have their aberrancy points on the y-axis. By taking the 3rd derivative, we find that the general quartic equation has one aberrancy point at $x_a = -\frac{b}{4a}$

So we set $x = x' - \frac{b}{4a}$

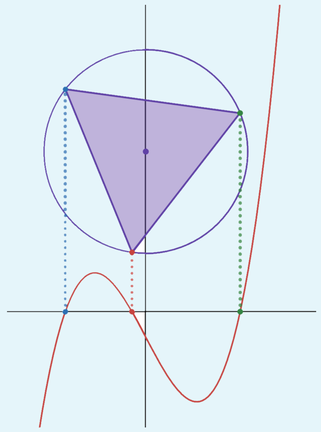
This equation in x' has the new coefficients: p, q and s; expressed in terms of the coefficients of the general form.

Once the equation in x' is solved, we can back-substitute the expressions for p, q and s, and shift the roots "back" by subtracting $-\frac{b}{4a}$ from the solutions we obtained.



Computer Demo 1

Solving the Cubic with a Triangle



Computer Demo 2

It is always possible to draw an equilateral triangle in the xy-plane so that the vertices line up with the solutions of the cubic equation.

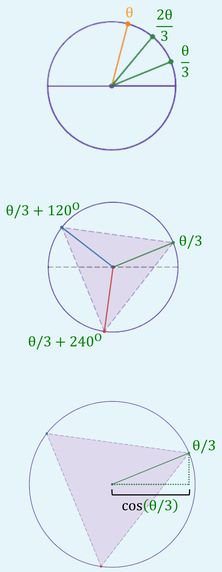
The radius of the circumcircle is $2\sqrt{-\frac{p}{3}}$, which determines the size.

The 3 angles the points form can be thought of in terms of trisecting an angle, θ.

This is because these angles are offset by 120° , and when added together 3 times, all of them form the same combined angle, θ.

To find the x-values themselves, we can use standard trigonometry. We take the cosine of the 3 angles, and scale them by the radius, giving

$$x = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta + 2\pi k}{3}\right)$$
$$k = 0,1,2$$



The Resolvent Quadratic & Complex Numbers

In the cubic formula, an equation called *the resolvent quadratic* is used to find the solution.

This quadratic expressed in an auxiliary variable, t, and is

$$t^2 + qt - \frac{p^3}{27} = 0$$

To solve the cubic, we find the 2 solutions of the quadratic in t; r_1 and r_2 and add their cube roots together, $x = \sqrt[3]{r_1} + \sqrt[3]{r_2}$

However, in the case of a cubic with 3 real solutions, the resolvent quadratic has complex roots! Therefore, r_1 and r_2 are a complex conjugate pair.

Each of the solutions has magnitude of $\left(-\frac{p}{3}\right)^{3/2}$ And the angle is the exact same angle θ that is being trisected in the real valued triangle!

When we extract the cube roots for r_1 and r_2, they form 2 equilateral triangles in the complex plane, rotated by an angle of $\theta/3$.

The magnitude of all 6 numbers is simply the cube root of the magnitude of $r_{1,2}$, which is $\sqrt{-\frac{p}{3}}$

There are nine combinations to add the cube roots in pairs, but we only want real numbers as answers.

Therefore, we must group the 3 complex conjugate pairs together to cancel out the imaginary components

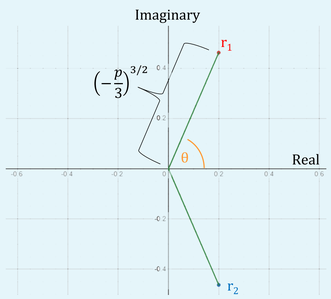
The angles of the cube roots are simply: $[\ \frac{1}{3}\theta;\ \ \frac{1}{3}\theta + \frac{2\pi}{3};\ \ \frac{1}{3}\theta + \frac{4\pi}{3}\ ]$

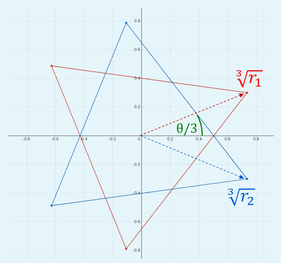
The real components are the magnitude times the cosine of the angle

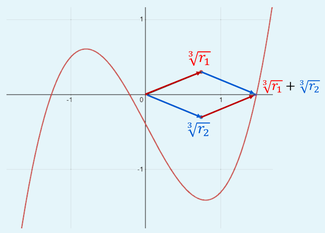
$$\mathrm{Re}\left(\sqrt[3]{r_{1,2}}\right) = \sqrt{-\frac{p}{3}}\cos\left(\frac{\theta+2\pi k}{3}\right)$$

When we sum each pairs, the imaginary components cancel out, and the real part is doubled. Therefore, our solution to the cubic is:

$$x = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta+2\pi k}{3}\right),\ k = 0,1,2$$





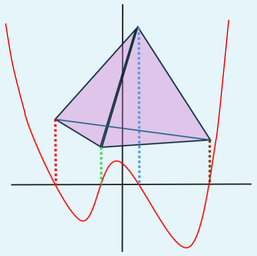


Solving the Quartic with a Tetrahedron

For the quartic equation, it is always possible to draw a tetrahedron in 3-D space so the vertices line up with the x coordinates of the roots.

Similar to how the resolvent quadratic can solve a cubic equation, a resolvent cubic can find solutions of a quartic equation.

Quartic Equation	Resolvent Cubic
$x^4 + px^2 + qx + s = 0$	$t^3 + \frac{p}{2}t^2 + \frac{p^2-4s}{16}t - \frac{q^2}{64} = 0$



Computer Demo 3

We already have a method to solve the cubic and obtain 3 real solutions.

Taking the square roots of the 3 solutions and adding them together gives the solutions to the quartic:

$x = \pm\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3}$ (there are 8 sign combinations, but only 4 solutions)

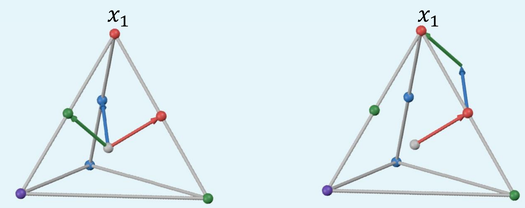
The Resolvent Cubic and Midpoints

The tetrahedron has 6 edges, and the midpoints of these edges relate to the solutions of the resolvent cubic.

The midpoints' x-coordinates line up with the square roots of r_1, r_2 or r_3, of which there are 6.

The geometric properties of the midpoints can reveal the x-coordinates of the vertices.

For example, consider midpoints of the 3 edges leading to the vertex x_1. Adding together the 3 vectors from the centroid to these midpoints yields a vector pointing directly at the vertex x_1



This shows geometrically how adding the square roots of the resolvent cubic solutions solves our quartic, since the x-coordinate of the tetrahedron is a solution to the original quartic.

Furthermore, the tetrahedron visualization show that only 4 combinations yield valid solutions

The vector addition only leads to a vertex when choosing 3 midpoints adjacent to a common vertex. There are only 4 valid combinations, corresponding to the 4 vertices. The other combinations are extraneous and do not lead to vertices nor solutions.

Sources and Acknowledgements

R. W. D. Nickalls, The quartic equation: alignment with an equivalent tetrahedron. Math. Gaz., 96 (March, 2012), pp. 49–55.

R. W. D. Nickalls, The quartic equation: invariants and Euler's solution revealed. Math. Gaz., 93 (March 2009), pp. 66–75.

Wikipedia contributors. "Cubic equation." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 13 Apr. 2026. Web. 13 Apr. 2026.

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