Completing the Square in 3D

Background

- Completing the square is an algebraic technique for solving quadratic equations
- A 3 dimensional analogue can solve various types of cubic equations
- I developed a expanded method to geometrically solve the general cubic equations in 3 dimensions

Completing the Square

Solving Quadratics

$$x^2 + bx + c = 0$$

Set x^2 as a square

bx as a rectangle

Add y to each side for $(x+y) \times (x+y)$ square

Combine $x \times y$ rectangles

Set $b = 2y \;\therefore\; b/2 = y$

Complete the square adding square $b/2 \times b/2$

In algebraic terms:

$$x^2 + bx + c = 0$$

$$x^2 + bx + \left(\frac{b}{2}\right)^2 + c = \left(\frac{b}{2}\right)^2$$

$$\underbrace{\left(x + \frac{b}{2}\right)^2}_{\text{our square}} + \underbrace{c = \left(\frac{b}{2}\right)^2}_{\text{constants}}$$

Which solves to

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

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Completing the Cube - *Depressed Cubics*

$$x^3 + px + q = 0$$

Terms of the perfect cube identity can represent shapes in an $x+y$ cube

$$(x+y)^3 = x^3 + 3yx^2 + 3y^2x + y^3$$

factoring $3yx$ $\quad (x+y)^3 = x^3 + 3yx(x+y) + y^3$

define $s = x + y$ $\quad (x+y)^3 = x^3 + 3xys + y^3$

geometrically

match to original equation

$$(x+y)^3 = x^3 + 3ys\,x + y^3$$
$$0 = x^3 + p\,x + q$$

① $p = 3ys$

Complete the Cube add y^3 to original equation

$$y^3 = x^3 + px + y^3 + q$$
$$y^3 = \underbrace{x^3 + 3yx^2 + 3y^2x + y^3}_{s^3} + q$$

② $y^3 = s^3 + q$

Solvable 2 Equation system

$$\left.\begin{array}{l} ① \; p = 3ys \\ ② \; y^3 = s^3 + q \end{array}\right\} \Rightarrow y^3 = \left(\frac{p}{3y}\right)^3 + q$$

$$y^3 = \frac{p^3}{27y^3} + q \quad \Rightarrow \quad y^6 = \frac{p^3}{27} + qy^3$$

Solve as quadratic of (y^3) $\quad (y^3)^2 - q(y^3) - \frac{p^3}{27} = 0$

$$y^3 = \frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \quad \Rightarrow \quad y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

Note: + and - solutions give same results

Substituting $s = \frac{p}{3y}$ yields $s = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$

Finally, using $x = s - y$ gives the formula for x



$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

Completing the Cube Twice - *General Cubics*

$$x^3 + bx^2 + cx + d = 0$$

Start with a cube with side length (x+y) then extend sides by length z

$$[(x+y)+z]^3 = (x+y)^3 + 3z(x+y)^2 + 3z^2(x+y) + z^3$$

$$(x+y+z)^3 = (x+y)^3 + \{3z(x+y)\}\cdot\{(x+y)+z\} + z^3 \quad \text{factor } 3z(x+y)$$

$$w^3 = (x+y)^3 + 3zw(x+y) + z^3 \quad \text{define } w = (x+y+z)$$

$$w^3 = x^3 + 3yx^2 + 3y^2x + y^3 + 3zwx + 3zwy + z^3 \quad \text{expand}$$

$$w^3 = [x^3] + 3y[x^2] + (3y^2 + 3zw)[x] + y^3 + z^3 + 3zwy \quad \text{group like terms}$$

$$w^3 = x^3 + 3y\,x^2 + (3y^2 + 3zw)\,x + y^3 + z^3 + 3zwy$$
$$0 = x^3 + b\,x^2 + c\,x + d \quad \text{match with original equation}$$

① $b = 3y$ $\quad$ ② $c = 3y^2 + 3zw$

$$[y^3 + z^3 + 3zwy] = x^3 + bx^2 + cx + [y^3 + z^3 + 3zwy] + d \quad \text{add to complete the cube}$$

$$y^3 + z^3 + 3zwy = \underbrace{x^3 + (3y)x^2 + (3y^2 + 3zw)x + y^3 + z^3 + 3zwy}_{w^3} + d \quad \text{substitute } b \,\&\, c$$

③ $y^3 + z^3 + 3zwy = w^3 + d$

Solvable 3 Equation system

$$\left.\begin{array}{l} ① \; b = 3y \\ ② \; c = 3y^2 + 3zw \\ ③ \; y^3 + z^3 + 3zwy = w^3 + d \end{array}\right\}$$

① $\Rightarrow y = \frac{b}{3}$ *Solved!*

Substitute y: ③ $\Rightarrow w^3 + d = \frac{b^3}{27} + z^3 + 3zwy$; ② $\Rightarrow c = \frac{b^2}{3} + 3zw$

Substitute w: $\frac{(c - \frac{b^2}{3})^3}{27z^3} + d = \frac{b^3}{27} + z^3 + 3zwy$; $3zw = c - \frac{b^2}{3}$

$w = \frac{c - \frac{b^2}{3}}{3z}$ *Solved!*

$$\frac{(c - \frac{b^2}{3})^3}{27z^3} + d = \frac{b^3}{27} + z^3 + \frac{bc}{3} - \frac{b^3}{9} \quad \text{Substitute } w \,\&\, y$$

$$\frac{(c - \frac{b^2}{3})^3}{27} + dz^3 = z^6 + \left(\frac{bc}{3} - \frac{2b^3}{27}\right)z^3 \quad \text{Multiply by } z^3$$

$$[z^3]^2 + \left(-\frac{2b^3}{27} + \frac{bc}{3} - d\right)[z^3] - \frac{\left(c - \frac{b^2}{3}\right)^3}{27} = 0$$

Solve as a Quadratic of $[z^3]$

$$z = \sqrt[3]{\left(\frac{b^3}{27} - \frac{bc}{6} + \frac{d}{2}\right) + \sqrt{\left(\frac{b^3}{27} - \frac{bc}{6} + \frac{d}{2}\right)^2 + \left(-\frac{b^2}{9} + \frac{c}{3}\right)^3}}$$

Combine solutions for z and w

$$w = \sqrt[3]{\left(-\frac{b^3}{27} + \frac{bc}{6} - \frac{d}{2}\right) + \sqrt{\left(\frac{b^3}{27} - \frac{bc}{6} + \frac{d}{2}\right)^2 + \left(-\frac{b^2}{9} + \frac{c}{3}\right)^3}}$$

Combining solutions using $x + y + z = w$

$$x = \sqrt[3]{\left(-\frac{b^3}{27} + \frac{bc}{6} - \frac{d}{2}\right) + \sqrt{\left(\frac{b^3}{27} - \frac{bc}{6} + \frac{d}{2}\right)^2 + \left(-\frac{b^2}{9} + \frac{c}{3}\right)^3}} + \sqrt[3]{\left(-\frac{b^3}{27} + \frac{bc}{6} - \frac{d}{2}\right) - \sqrt{\left(\frac{b^3}{27} - \frac{bc}{6} + \frac{d}{2}\right)^2 + \left(-\frac{b^2}{9} + \frac{c}{3}\right)^3}} - \frac{b}{3}$$

The Cubic Formula

Completing the Square, in 4D?

Can we extend this to Quartic equations using x^4? Here is what I know so far:

- Methods exist for solving any quartic equation using radicals, but are not a direct extensions of completing the square
- 4-D geometry is extremely difficult to visualize, but an algebraic solution would be possible by rearranging and matching terms similar to the cubic
- Every algebraic solution to the quartic requires solving a cubic as part of the solution, much like how cubic solutions use the quadratic formula
- I have tried to solve this algebraically, but the resulting equations are quite complicated, but I would encourage you to try this yourself!

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

Sources

- "How Imaginary Numbers Were Invented" - Veritasium
- "500 Years of Not Teaching the Cubic Formula" - Mathologer
- Wikipedia– Cubic Equation

Further Topics

- Ferrari's and Descartes' methods for solving quartic equations
- Trigonometric Solutions: both Algebraic and Geometric
- Complex solutions to cubics
- The case of 3 real solutions
- Completing the Hypercube

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